Finite Element Spaces

Triangulation and Finite Element Spaces

Definition: Let \( \Omega \) be a domain (connected open set) in \( \mathbb{R}^n \) with Lipschitz-continuous boundary \( \Gamma = \partial \Omega \). A triangulation \( T \) of \( \Omega \) is a finite collection of compact sets \( \{ K_i \} \) s.t.:

1. \( \bigcup_{K \in T} K = \overline{\Omega} \)
2. \( K' \neq \phi \) for \( K \in T \)
3. \( \partial K \) is Lipschitz-continuous for \( K \in T \)
4. If \( K_1, K_2 \in T \) and \( K_1 \neq K_2 \) then \( K_1 \cap K_2 = \phi \)
5. The boundaries \( \partial K \) can be partitioned into a finite number of faces \( F \) s.t. for any \( K \in T \) and any face \( f \in F \) corresponding to \( K \) (i.e., \( f \subset \partial K \)), then either \( f \subset \Gamma \) called an exterior boundary face or there is exactly one \( K' \in T \), \( K' \neq K \) s.t. \( f \subset \partial K' \) called an interior boundary face.

Definition: A finite-element space is a triplet \((\Omega, T, V)\) consisting of a domain \( \Omega \), a triangulation \( T \) of \( \overline{\Omega} \) and a finite-dimensional subspace \( V \) of functions on \( \overline{\Omega} \). The finite-dimensional spaces \( P_k \) are the restrictions of the functions in \( V \) to the subset \( K \in T \) i.e:

\[
P_k = \{ v|_K : v \in V \} \tag{1.1}
\]

Theorem: If \( P_k \subset H^1(K) \) for all \( K \in T \) and if \( V \subset C^0(\overline{\Omega}) \), then:
1. \( V \subset H^1(\Omega) \)
2. \( V_0 = \{ v \in V : v = 0 \text{ on } \Gamma \} \subset H^1_0(\Omega) \)

proof: Let \( v \in V \). It is required to find a function \( v_i \in L^2(\Omega) \) s.t.

\[
\forall \phi \in D(\Omega), \int_{\Omega} v_i \phi dx = -\int_{\Omega} v \phi dx \tag{1.2}
\]

To this end, let \( v_i \) be the function (in \( L^2(\Omega) \)) s.t. \( v_i|_K = \partial_i(v|_K) \). Using Green’s formula, we have for each \( K \in T \):

\[
\int_K \partial_i(v|_K) \phi dx = -\int_K v|_K \partial_i \phi dx + \int_{\partial K} v|_K \phi v_{i,k} d\gamma \tag{1.3}
\]

Here, \( v_{i,k} \) is the \( i \)-th component of the unit outer normal along \( \partial K \). Summing over \( K \in T \) we have:

\[
\int_{\Omega} v_i \phi dx = -\int_{\Omega} v \phi dx + \sum_{K \in T} \int_{\partial K} v|_K \phi v_{i,k} d\gamma
= -\int_{\Omega} v \phi dx \quad \text{as } v|_K \text{ is continuous and } \phi = 0 \text{ on } \Gamma. \tag{1.4}
\]
Finite Elements

Definition: A finite element of $\mathbb{R}^n$ is a triplet $(K, P, \Sigma)$ where:

1. $K$ is a compact subset of $\mathbb{R}^n$ with Lipschitz-continuous boundary and non-empty interior.
2. $P$ is a finite-dimensional space of real-valued functions on $K$.
3. $\Sigma$ is a finite, unisolvent, linearly independent set of linear functionals $\{\phi_i\}_{i=1}^N$ on $P$ called the degrees of freedom of the element. The basis functions of the finite element are the functions $p_i \in P$, $i = 1, \ldots, N$ s.t. $\phi_i(p_j) = \delta_{ij}$.

Definition: If $(K, P, \Sigma)$ is a finite element and $v$ is a (sufficiently smooth) function defined on $K$, then the $P$-interpolant of $v$ is given as:

$$\Pi(v)(x) = \sum_{i=1}^N \phi_i(v) p_i(x) \quad (1.5)$$

Note: $v \in P \iff \Pi(v) = v$ and the $P$-interpolant of $v$ is characterized by the conditions that $v \in P$ and $\phi_i(\Pi(v)) = \phi_i(v)$ for $i = 1, \ldots, N$.

Definition: Two finite elements $(K, P, \Sigma)$ and $(L, Q, \Xi)$ are said to be equal if:

$$K = L, \quad P = Q \quad \text{and} \quad \Pi_K = \Pi_L \quad (1.6)$$

Definition: If $(K, P, \Sigma)$ is a finite element and if the degrees of freedom $\phi \in \Sigma$ are of the form:

a) $\phi: p \rightarrow p(a^0_i)$

b) $\phi: p \rightarrow Dp(a^1_i)\xi^1_i$

c) $\phi: p \rightarrow D^2(p^2_j)(\xi^2_{ik}, \xi^2_{il})$ \quad (1.7)

where the points $a^r_i, r = 0, 1, 2$ belong to $K$ and are called the nodes $N_K$ of the element, and the vectors $\xi^1_i, \xi^2_{ik}, \xi^2_{il}$ are either fixed vectors in $\mathbb{R}^n$ or else are constructed from the geometry of $K$, then the finite element is said to be nodal-based. If the degrees of freedom are exclusively of type a), $p \rightarrow p(a^0_i)$, the element is said to be a Lagrange finite element; otherwise the element is said to be a Hermite element.

Affine Equivalence

Definition: Two nodal-based finite elements $(\hat{K}, \hat{P}, \hat{\Sigma})$ and $(K, P, \Sigma)$ are said to be affine-equivalent if there exists a 1-1 affine mapping:
\( F: \hat{K} \rightarrow K \) s.t. \( F(\hat{x}) = B\hat{x} + b \) for which \( B \) is linear and invertible and s.t.:

\( a) \quad K = F(\hat{K}) \)

\( b) \quad P = \{ p: K \rightarrow \mathbb{R} : p = \hat{p} \circ F^{-1} \text{ and } \hat{p} \in \hat{P}_1 \} \quad (1.8) \)

\( c) \quad a_i' = F(a_i') \quad r = 0,1,2 \)

\( d) \quad \xi_i = Bz_i, \quad \xi_i^2 = Bz_i^2, \quad \xi_i^3 = Bz_i^3 \)

**Theorem:** Let \((\hat{K}, \hat{P}, \hat{\Sigma})\) and \((K, P, \Sigma)\) be two nodal-based, affine-equivalent finite elements where \( F: \hat{K} \rightarrow K \) and \( \nu = \hat{\nu} \circ F^{-1} \) for \( \hat{\nu} \in \text{dom}\hat{\Pi} \) and \( \nu = \hat{\nu} \circ F^{-1} \in \text{dom}\Pi \).

Then if \( \{ \hat{p}_i \}_{i=1}^N \) are the basis functions of \( \hat{K} \) and \( \{ p_i \}_{i=1}^N \) are the basis functions of \( K \), then the interpolation operators \( \hat{\Pi} \) and \( \Pi \) are related by:

\[ \hat{\Pi} \nu = \hat{\Pi} \hat{\nu} \quad (1.9) \]

**proof:** \( \Pi \nu = \sum_i \nu(a_i^0)p_i^0 + \sum_{i,k} (D\nu(a_i^1)\xi_{ik})p_i^1 + \sum_{i,j,k} (D^2\nu(a_i^2)(\xi_{ik}^2, \xi_{ij}^2))p_{ikl}^2 \)

However, from the formula for the derivative of function composition, we have:

\[ D\nu(a_i^1)\xi_{ik}^1 = D\nu(F(\hat{a}_i^1))Bz_{ik}^1 = D\nu(F(\hat{a}_i^1))DF(\hat{a}_i^1)z_{ik}^1 = D(\nu \circ F)(\hat{a}_i^1)z_{ik}^1 \]

\[ D^2\nu(a_i^2)(\xi_{ik}^2, \xi_{ij}^2) = D^2\hat{\nu}(\hat{a}_i^2)(\hat{z}_{ik}^2, \hat{z}_{ij}^2) \text{ and similarly} \quad (1.10) \]

Then the result follows immediately.

**Note:** In general, when the finite elements are not necessarily nodal-based, a necessary and sufficient condition for (1.9) is that:

\[ \hat{\phi}(\hat{\nu}) = \phi(\nu) \quad i = 1, \ldots, N \quad (1.11) \]